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www.emeraldinsight.com/0961-5539.htm

Received April 2003
Revised September 2003
Accepted December 2003

Easy and rapid computation of the transfer of heat from annular fins of nearly optimal profile with the finite-difference technique and the shooting method

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Keywords *Heat transfer, Numerical analysis, Optimization techniques*

Abstract *It is undeniable that the annular fin of hyperbolic profile with constant thermal conductivity and uniform convective coefficient is important in many applications of heat transfer engineering. The importance of this fin configuration stems from its close resemblance to the annular fin of optimal cross section capable of delivering maximum heat transfer for a given volume of material. This paper addresses two simple numerical procedures for solving the generalized Bessel equation that governs the temperature variation in annular fins of hyperbolic profile, one is the finite-difference technique with an uncharacteristic coarse mesh and the other is the shooting method. Certainly, the central objective here is to avoid the evaluation of the elegant, but intricate exact analytic temperature distributions and companion fin efficiencies containing modified Bessel functions of fractional order.*

Nomenclature

A_p	= profile area, m^2	Q_{ideal}	= ideal heat transfer rate, W
c	= normalized radii ratio, r_1/r_2	Q_{ref}	= reference heat transfer rate, W
h	= convective coefficient, W/m^{2}K	r	= radial coordinate, m
H^2	= thermo-geometric parameter, $h(1/\delta_1 r_1)/k$, $1/m$	r_1	= inner radius, m
H_ν (*)	= modified Bessel function of second kind and order ν	r_2	= outer radius, m
I_ν (*)	= modified Bessel function of first kind and order ν	R	= dimensionless r, r/r_2
k	= thermal conductivity, W/m K	T	= temperature, K
M^2	= dimensionless H^2 or modified Biot number, $H^2 r_2^3$	T_b	= base temperature, K
Q	= heat transfer rate, W	T_t	= tip temperature, T (r_2), K
		T_∞	= fluid temperature, K
		w	= length, $r_2 - r_1$, m
		$y(r)$	= profile function $t\ (r_1/r)$, m

International Journal of Numerical Methods for Heat & Fluid Flow
Vol. 14 No. 8, 2004
pp. 1002-1010

0961-5539
DOI 10.1108/09615530410557414

z	= dimensionless temperature gradient, $d\theta/dR$	θ	= dimensionless T, $(T - T_\infty)/(T_b - T_\infty)$
δ_1	= inner semi-thickness, m	σ	= alternative dimensionless heat loss, equation (12)
δ_2	= outer semi-thickness, m		
η	= fin efficiency, Q/Q_{ideal}		

Introduction

Annular finned tubes are commonly used in a variety of liquid-gas heat exchange devices with the purpose of augmenting the transfer of heat from primary tube surfaces to adjacent gas streams (Hewitt *et al.*, 1993; Kraus *et al.*, 2000; Webb, 1994). Typical applications of annular finned tubes are found in air-cooled engines of motorcycles and automobiles, HVAC systems, etc.

In modern era, heat exchange devices are becoming increasingly sophisticated and continually require greater precision, adequate sizing, improved reliability, and extended life (Hewitt *et al.*, 1993; Kraus *et al.*, 2000; Webb, 1994). To meet these stringent demands, exceptional fin profiles have been ceaseless explored in theoretical studies, numerical simulations and experimental measurements all over the world (Hewitt *et al.*, 1993; Kraus *et al.*, 2000; Webb, 1994). In this regard, the annular fin of hyperbolic profile turns out to be the foremost important fin that can be attached to round tubes because it resembles the optimal annular fin of convex parabolic profile discovered by Schmidt (1926). Unquestionably, the latter has become staple in heat transfer engineering because of its unique ability to reject maximum heat transfer for a given volume of metallic material (Hewitt *et al.*, 1993; Kraus *et al.*, 2000; Webb, 1994).

From a fundamental standpoint, the temperature change along an annular fin of hyperbolic profile with constant thermal conductivity and uniform convective coefficient is governed by a two-term differential equation of second-order with a variable coefficient. By virtue of a proper transformation, the differential equation falls under the category of a generalized Bessel equation. Although this equation admits an exact analytical solution, it is of intricate form because of the presence of modified Bessel functions of fractional order. Hence, the numerical evaluation of temperatures and/or heat transfer rates is quite complicated and time-consuming.

Setting aside the use of modified Bessel functions deliberately, this paper addresses two simple numerical procedures for solving the one-dimensional fin equation that governs the annular fin of hyperbolic profile. The two procedures are the finite-difference technique with an uncharacteristic coarse mesh and the shooting method. In the former, small systems of algebraic equations were solved with the elimination of unknowns by hand, whereas large systems if necessary could be solved with the Gauss elimination method. In the latter, a fourth-order Runge-Kutta integration algorithm was paired with a standard linear interpolation formula for solving the system of two differential equations of first-order. All numerical calculations were carried out with the symbolic algebra software Maple (Redfern, 1996) on a personal computer.

Generalized Bessel equation

Figure 1 shows the path of two symmetric hyperbola $y(r) = \delta(r_1/r)$ bounding a tapered annular fin that features four dimensions: the inner radius r_1, the inner

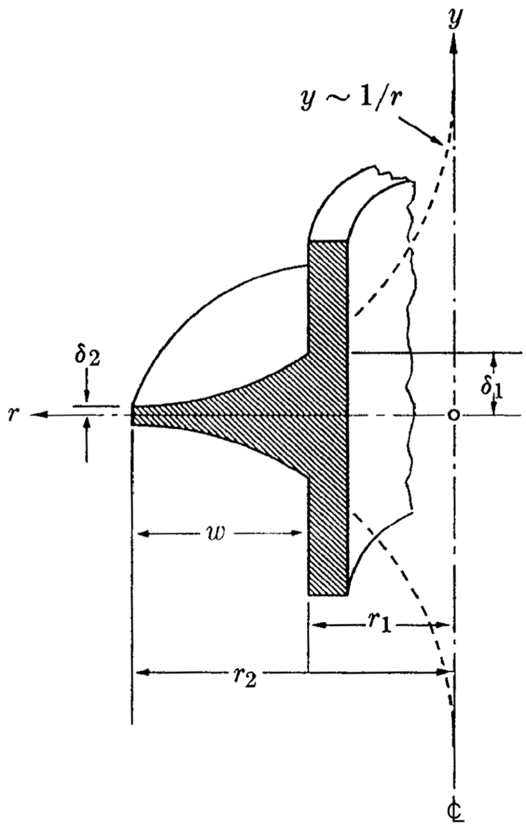

Figure 1.
Sketch of an annular fin of hyperbolic profile

semi-thickness δ_1, the outer radius r_2 and the outer semi-thickness δ_2. Schneider (1955) stated that this kind of fin configuration, appropriately called the annular fin of hyperbolic profile, is of remarkable notoriety because of its close resemblance to the optimal annular fin of convex parabolic profile, which is capable of delivering maximum heat transfer for a given volume of material (Schmidt, 1926). Unquestionably, the pitfall of the optimal annular fin of convex parabolic profile is its sharp tip because it may jeopardize the safety of technical personnel working in plant environments.

Under the assumption of constant thermal conductivity k and uniform convective coefficient h, the transfer of heat from an annular fin of hyperbolic profile to a surrounding fluid is modeled by the dimensionless fin equation (Schneider, 1955):

$$R^2\frac{\mathrm{d}^2\theta}{\mathrm{d}R^2} - M^2R^3\theta = 0 \tag{1}$$

where the normalized dimensionless variables for the temperature θ and radial variable R are

$$\theta = \frac{T - T_\infty}{T_b - T_\infty}, \quad R = \frac{r}{r_2} \tag{2}$$

The boundary conditions imposed on equation (1) are prescribed temperature at the fin base

$$\theta = 1, \quad R = c \tag{3a}$$

and negligible heat loss at the fin tip

$$\frac{d\theta}{dR} = 0, \quad R = 1 \tag{3b}$$

In view of the foregoing, two parameters characterize the temperature variation along the annular fin of hyperbolic profile. One is the thermo-geometric parameter $M^2 = h(r_2^3/\delta_1 r_1)/k$ appearing in equation (1), which fundamentally speaking may be viewed as an enlarged Biot number. The other is a geometric parameter, the normalized radii ratio $0 < c = r_1/r_2 \leq 1$ surfacing up in equation (3a).

The heat transfer rate Q from a fin to a neighboring fluid is customarily computed in an indirect manner with the fin efficiency $\eta = Q/Q_{\text{ideal}}$. Hence, two avenues are possible:

(1) utilizing the derivative of $\theta(R)$ at the fin base:

$$\eta = \frac{Q}{Q_{\text{ideal}}} = \frac{-2 \left.\frac{d\theta}{dR}\right|_{R=c}}{M^2(1 - c^2)} \tag{4}$$

or

(2) employing the integral of $\theta(R)$ over the fin length:

$$\eta = \frac{Q}{Q_{\text{ideal}}} = \frac{2\int_c^1 \theta R \, dR}{(1 - c^2)} \tag{5}$$

Solution procedures

With a proper transformation, equation (1) falls under the category of a generalized Bessel type equation (Arpaci, 1966).

Exact analytical method

The exact analytic solution of equations (1) and (2) taken from Schneider (1955), gives way to the dimensionless temperature distribution $\theta(R)$:

$$\theta(R) = \sqrt{\frac{R}{c}} \left[\frac{I_{1/3}\left(\frac{2}{3}MR^{3/2}\right) I_{2/3}\left(\frac{2}{3}M\right) - I_{-1/3}\left(\frac{2}{3}MR^{3/2}\right) I_{-2/3}\left(\frac{2}{3}M\right)}{I_{1/3}\left(\frac{2}{3}Mc^{3/2}\right) I_{2/3}\left(\frac{2}{3}M\right) - I_{-1/3}\left(\frac{2}{3}Mc^{3/2}\right) I_{-2/3}\left(\frac{2}{3}M\right)} \right] \tag{6}$$

Similarly, the exact analytic fin efficiency η taken from Schneider (1955) is:

$$\eta = \frac{2c^{1/2}}{M(1-c^2)}\left[\frac{I_{-2/3}\left(\frac{2}{3}Mc^{3/2}\right)I_{2/3}\left(\frac{2}{3}M\right) - I_{2/3}\left(\frac{2}{3}Mc^{3/2}\right)I_{-2/3}\left(\frac{2}{3}M\right)}{I_{-1/3}\left(\frac{2}{3}Mc^{3/2}\right)I_{-2/3}\left(\frac{2}{3}M\right) - I_{1/3}\left(\frac{2}{3}Mc^{3/2}\right)I_{2/3}\left(\frac{2}{3}M\right)}\right] \tag{7}$$

where I_ν (*) is the modified Bessel function of first kind and order ν and H_ν (*) is the modified Bessel function of second kind and order ν.

To quantify the heat transfer rate Q from the annular fin of hyperbolic profile, it becomes evident that the evaluation of equation (7) involving modified Bessel function of fractional order is a laborious task even with the help of a symbolic algebra code, like Maple (Redfern, 1996). On the other hand, numerical values of the fin efficiency may be approximately read from the fin efficiency diagram in Schneider (1955). Surprisingly, only three curves for the radii ratios: $c = 1$, 1/2 and 1/4 are portrayed in the fin efficiency diagram (Schneider, 1955). Therefore, for other radii ratios contained in the realistic interval $1/4 \leq c \leq 1$, the interpolation between points in the three curves turns out to be tedious and somehow inaccurate.

Finite-difference technique

The fin region $c \leq R \leq 1$ is divided into I equal intervals of size $\Delta R = (1-c)/I$. Employing the central formulation for the derivative of second-order with a truncation error of order $(\Delta R)^2$ (Hildebrand, 1988), the differential equation (1) is converted into the following finite-difference equation

$$\theta_{i+1} - [2 + M^2(\Delta R)^2 R_i]\theta_i + \theta_{i-1} = 0 \tag{8}$$

where $i = 0, 1, 2, \ldots, I$.

The first boundary condition in equation (3a) gives way to $\theta_0 = 1$ in equation (8). For the second boundary condition in equation (3b), the central formulation with a truncation error of order $(\Delta R)^2$ sets off the equality $\theta_{I+1} = \theta_{i-1}$ in equation (8). With these additions, equation (8) furnishes a system of I algebraic equations for the I node temperatures θ_I where $i = 1, 2, \ldots, I$.

Shooting method

The shooting method is a numerical procedure that consists in a mathematical transformation of a well posed two-point boundary value problem into an equivalent, but incomplete initial value problem (Hildebrand, 1988).

Letting $\mathrm{d}\theta/\mathrm{d}R = z$, the second-order differential equation (1) is transformed into the following system of two differential equations of first-order:

$$\frac{\mathrm{d}\theta}{\mathrm{d}R} = z \tag{9}$$

$$\frac{\mathrm{d}z}{\mathrm{d}R} = -M^2 R\theta \tag{10}$$

in which the dependent variables are the temperature θ and the temperature gradient z.

Conceptually, the proper specification of an initial value problem in the R-domain $[c, 1]$ demands the application of two initial conditions at the left extreme $R = c$. One initial condition for θ is obtained from equation (3a), whereas the other initial condition for z is missing and must be guessed up front. The satisfaction of the far away boundary condition, namely $d\theta/dR = 0$ at the right extreme $R = 1$, needs to be fulfilled. In other words, this statement signifies that the dimensionless temperature gradient $z(1) = 0$ in compliance with equation (3b).

Representative test case

An annular fin of hyperbolic profile envelops a round metallic tube of given radius r_1. Heat is rejected from the annular fin to a neighboring fluid at a temperature T_∞. The fin base has a fixed temperature T_b and the heat loss through the fin tip is considered insignificant. The fin is constructed with a metal of constant thermal conductivity k in the temperature interval of operation $T_\infty \leq T \leq T_b$. Assuming that the convective coefficient h is uniform, the goal is to calculate the temperature distribution and the heat transfer rate using two numerical procedures: the finite-difference technique and the shooting method.

To demonstrate the feasibility of the two powerful numerical procedures, it suffices to study one typical fin/fluid assembly only. For instance, let us select a fin/fluid assembly characterized by a normalized radii ratio $c = 1/4$ and a modified Biot number $M^2 = 6.572$, which constitutes a reasonable point of reference. The exact fin efficiency $\eta = 0.523$ was pulled out from the fin efficiency diagram in Schneider (1955).

Finite-difference technique

The annular fin of hyperbolic profile is first divided into three equal intervals of size $\Delta R = 1/4$. With this coarse mesh, a system consisting of three algebraic equations relates to the three nodal temperatures θ_1, θ_2, θ_3:

$$\begin{aligned} &\text{at } i = 1, \quad 2.206\theta_1 - \theta_2 = 1 \\ &\text{at } i = 2, \theta_1 - 2.308\theta_2 + \theta_3 = 0 \\ &\text{at } i = 3, 2\theta_2 - 2.411\theta_3 = 0 \end{aligned} \tag{11}$$

Clearly, the above system being small can be easily solved by elimination by hand. The nodal temperatures with inherent truncation errors of order 1/16 are listed in the accompanying Table I.

Next, the numerical calculation of the fin efficiency η may be determined by hand too. For the numerical evaluation of the derivative of first-order in equation (4), a two-point forward and a three-point forward formulations are employed having respective truncation errors of order ΔR and $(\Delta R)^2$ (Hildebrand, 1988). Doing the algebra, the fin efficiencies turn out to be $\eta = 0.451$ for the two-point forward formulation and 0.540 for the three-point forward formulation. Subsequently, the

R	θ
$R_0 = 0.25$	$\theta_0 = 1$
$R_1 = 0.50$	$\theta_1 = 0.653$
$R_2 = 0.75$	$\theta_2 = 0.442$
$R_3 = 1$	$\theta_3 = 0.367$

Table I. Temperature distribution produced by the finite-difference technique for $c = 1/4$ and $M^2 = 6.572$

numerical integration of the integral I in equation (5) can be performed with the trapezoidal and Simpson's 3/8 rules whose respective truncation errors are of order ΔR and $(\Delta R)^2$ (Hildebrand, 1988). Once the algebra has been completed, the corresponding values of the fin efficiencies are $\eta = 0.516$ and 0.519. In view of this, it is evident that with the exception of the two-point forward formulation for the derivative of first-order, the accuracy of the other three numerically computed efficiencies stays within a few percent of the exact value of $\eta = 0.523$. In addition, it may be realized that the better three-point forward formulation for the derivative underpredicts the exact fin efficiency whereas the two integration approaches overpredict the exact fin efficiency.

From the framework of conventional thermal performance, annular fins of any cross section should operate with efficiencies near one; that is much greater than the 0.523 chosen here. In other words, this statement implies a subset of fin/fluid assemblies characterized by the tandem of parameters $c > 1/4$ and $M^2 < 6.572$. These assemblies can be safely analyzed with a coarse mesh related to small systems of algebraic equations. However, in the unlikely event that a fine mesh becomes necessary, large systems of algebraic equations may be solved with the Gaussian elimination method and the symbolic algebra software Maple (Redfern, 1996).

Shooting method

The nature of the mixed boundary conditions in equation (3a) and (3b) suggests that the temperature starts with $\theta = 1$ at the base $R = 1/4$. Thereafter, the temperature θ decreases monotonically with R, so that the temperature slope $z = \mathrm{d}\theta/\mathrm{d}R$ of the temperature curve $\theta(R)$ is always negative. In addition, it may be inferred that the largest temperature slope z occurs at the base $R = 1/4$, thereafter z decreases monotonically with R and eventually dies out to zero at the tip $R = 1$.

Theoretically speaking, to initiate the numerical integration of the system of equations (9) and (10), we must guess a value of the temperature slope at the base, i.e. $z(1/4)$. In practice, to speed up the calculations, first $z(1/4)$ may be computed from an akin straight fin with uniform cross section due to the same thermo-geometric parameter $M^2 = 6.572$. For this reference straight fin, the exact temperature distribution yields the initial temperature slope $z(1/4) = -0.988$.

Among the step sizes ΔR tested in the numerical experiments, the last two employed were $\Delta R = 0.01$ and 0.001. For these, the proper variations of the temperature distributions $\theta(R)$ and the temperature gradient distributions $\mathrm{d}\theta(R)/\mathrm{d}R$ are shown in Figure 2 along with their corresponding convergence patterns. The numerical integration of the system of equations (9) and (10) has been done with a fourth-order Runge-Kutta algorithm (Hildebrand, 1988) using the symbolic algebra software Maple (Redfern, 1996).

For the successful retrieval of the terminal boundary condition $z(\theta) = \mathrm{d}\theta(1)/\mathrm{d}R = 0$ as given in equation (3b), the conclusion drawn here is that the initial temperature slope $z(1/4) = \mathrm{d}\theta(1/4)/\mathrm{d}R$ must be equal to -1.61. Moreover, the optimal step size ΔR demanded for the accurate numerical integration of the system of equations (9) and (10) is 0.001.

Ultimately, inserting the accurate initial temperature slope $z(1/4) = \mathrm{d}\theta(1/4)/\mathrm{d}R = -1.61$ into equation (4) furnishes the fin efficiency $\eta = 0.519$ right away. This number compares favorably with the exact efficiency of 0.523 computed by evaluating the modified Bessel functions of fractional form in (Schneider, 1955). For completeness,

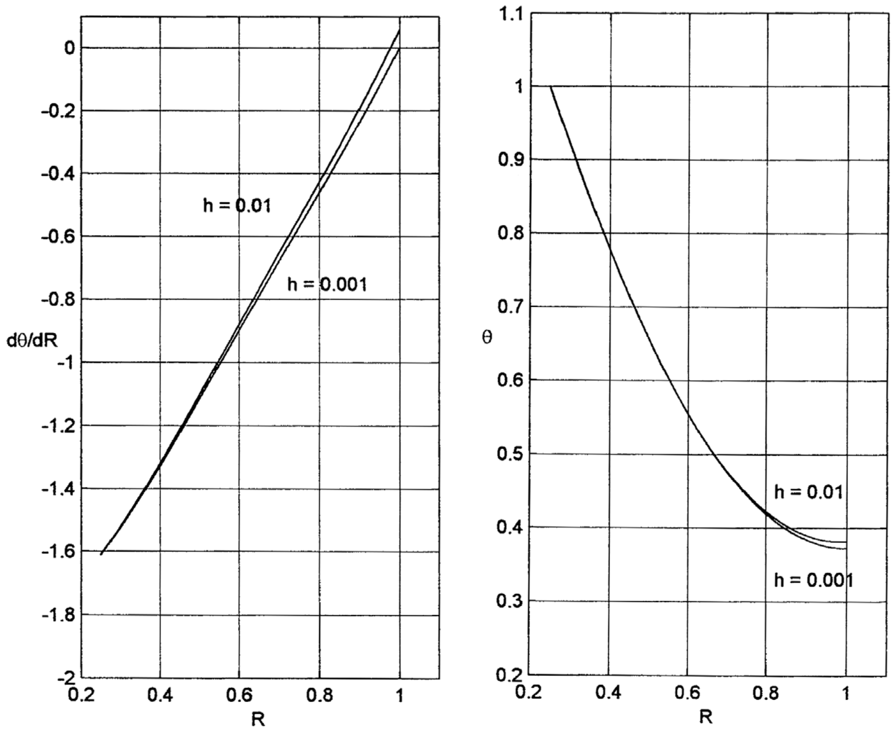

Figure 2. Convergence patterns exhibited by the temperature distribution $\theta(R)$ and the temperature gradient distribution $d\theta(R)/dR$ as calculated by the shooting method

Table II compares the accuracy of the fin efficiency delivered by the four different numerical procedures employed here.

Alternative fin efficiency

It is worth mentioning that Shamsundar (2002) has suggested that the fin efficiency concept is not useful when optimal dimensions are sought to maximize the heat loss for a given mass of material. For the particular case of an annular fin of hyperbolic profile, the alternative expression for the dimensionless heat loss σ is written as

$$\sigma = \frac{Q}{Q_{\text{ref}}} \tag{12}$$

where $Q_{\text{ref}} = 4\pi r_1 k(T_{\text{b}} - T_\infty)[A_{\text{p}}/2(h/k)^2]^{1/3}$. Thereby, for a given mass, the profile area of a hyperbolic annular fin, $A_{\text{p}} = 2\delta_1 r_1 \ln(r_2/r_1)$ is constant and σ delivers the

Table II. Comparison of the computed fin efficiencies for $c = 1/4$ and $M^2 = 6.572$

Exact	Derivative Two-point (error)	Derivative Three-point (error)	Integration Trapezoidal (error)	Integration 3/8 Simpson (error)
0.523	0.451 (−13.8 per cent)	0.540 (3.3 per cent)	0.516 (−1 per cent)	0.519 (−0.8 per cent)

corresponding maximum heat loss Q_{max} in a natural way. This physical-oriented approach articulates perfectly with the finite-difference technique and the shooting method employed here.

Conclusions

The utilization of simple numerical computational procedures fills a definitive gap in the accessible literature on fin heat transfer (Kraus *et al.*, 2000). Overall, it has been demonstrated that for the nearly optimal annular fin of hyperbolic profile with constant thermal conductivity k and uniform convective coefficient h, both finite-difference technique and shooting method are capable of estimating temperature distributions and heat transfer rates of good quality with minimal effort. It should be added that the finite-difference technique with its patented coarse meshes linked to small systems of algebraic equations in a reduced R-domain [c, 1] is more attractive than the shooting method. Further, it should be emphasized that the alternative expression for the dimensionless heat loss $\sigma = Q/Q_{ref}$ obtained from equation (12), seems to be superior than the traditional relation for the fin efficiency $\eta = Q/Q_{ideal}$ normally used in the heat conduction literature. In fact, the σ ratio elucidates the physics of the problem in a remarkable way.

Finally, this paper includes valuable material for two engineering audiences:

1. thermal design engineers engaged in real-life engineering practice; and
2. instructors of courses on heat transfer and/or heat exchangers in the mechanical and chemical engineering programs.

References

Arpaci, V. (1966), *Conduction Heat Transfer*, Addison-Wesley, Reading, MA.

Hewitt, G.F., Shires, G.L. and Bott, T.R. (1993), *Process Heat Transfer*, CRC Press, Boca Raton, FL.

Hildebrand, F.B. (1988), *Introduction to Numerical Analysis*, 2nd ed., Dover, New York, NY.

Kraus, A.D., Aziz, A. and Welty, J.R. (2000), *Extended Surface Heat Transfer*, Wiley, New York, NY.

Redfern, D. (1996), *The Maple Handbook*, Springer, New York, NY.

Schmidt, E. (1926), "Die Wärmeübertragung durch Rippen", *Zeitschrift des Vereines Deutscher Ingenieure*, Vol. 70, pp. 885-9, and 947-951.

Schneider, P.J. (1955), *Conduction Heat Transfer*, Addison-Wesley, Reading, MA.

Shamsundar, N. (2002), Personal communication, Department of Mechanical Engineering, University of Houston, Houston, TX, November.

Webb, R.L. (1994), *Principles of Enhanced Heat Transfer*, Wiley Interscience, New York, NY.